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The consistent equations of the theory of plane curvilinear rods for finite displacements and linearized problems of stability☆

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ABSTRACT

Based on a previously constructed, consistent version of the geometrically non-linear equations of elasticity theory, for small deformations and arbitrary displacements, and a Timoshenko-type model taking into account transverse shear and compressive deformations, one-dimensional equations of an improved theory are derived for plane curvilinear rods of arbitrary type for arbitrary displacements and revolutions and with loading of the rods by follower and non-follower external forces. These equations are used to construct linearized equations of neutral equilibrium that enable all possible classical and non-classical forms of loss of stability (FLS) of rods of orthotropic material to be investigated, ignoring parametric deformation terms in the equations. These linearized equations are used to find accurate analytical solutions of the problem of plane classical flexural-shear and non-classical flexural-torsional FLS of a circular ring under the combined and separate action of a uniform external pressure and a compression in the radial direction by forces applied to both faces.

Fundamental results associated with the construction of geometrically non-linear equations of the theory of elastic and inelastic rods for arbitrary displacements, and also with the study of their forms of loss of stability (FLS) under various types of conservative and non-conservative loads, were obtained as long ago as the middle of the twentieth century (see, for example, Ref. 1). The need for further and more extensive investigations of rod stability theory arose unexpectedly in the light of the results of recent studies.[2–4] According to these results, for small deformations, the use of kinematic relations in the quadratic approximation, that are well known in geometrically non-linear elasticity theory, and considered to be absolutely correct in all the scientific and academic literature, produces "false" bifurcation solutions under certain types of loading. For the case of small deformations, a consistent version was constructed, and the simplest examples of its application were considered, involving the reduction of the two-dimensional non-linear problem of deformation of a strip in the form of a rod to one-dimensional equations and their subsequent use to determine the possible FLS under characteristic types of loading. The fundamentally novel results that were obtained in this case concerned the FLS of rods under uniform transverse compression and pure shear[3] and also novel non-classical FLS of cylindrical shells for certain types of loading, which were based on the linearized equations of momentless shell theory.[4] These investigations were then continued[5] in the context of constructing consistent equations of the theory of thin shells for small deformations and arbitrary displacements, the identification of all possible FLS of a cylindrical shell under torsion on the basis of these equations, the construction of general linearized equations of elastic stability theory for rectilinear rods, and the investigation, on the basis of these, of all possible classical and non-classical FLS under different types of loading by conservative forces.

Developing the results described above, in the present paper for plane curvilinear rods of general form, based on Timoshenko's kinematic model, taking transverse shear and compressive deformations into account, consistent equations of the geometrically non-linear theory of plane curvilinear rods under arbitrary displacements are constructed, the use of which enables us to identify all their possible classical and non-classical FLS under the action of "dead" and "follower"[1] external forces.

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1. Kinematic relations

For the space occupied by the rod, we will assume the following parameterization

$$\mathbf{R}(x, y, z) = \mathbf{r}(x) + \boldsymbol{\rho}(x, y, z) = \mathbf{r}(x) + y\mathbf{n}(x) + z\mathbf{b}(x) \tag{1.1}$$

where $\mathbf{r}(x)$ is the equation of the centreline L relative to the natural parameter x, and $\mathbf{t} = \mathbf{r}' = d\mathbf{r}/dx$, $\mathbf{n}$ and $\mathbf{b}$ are unit vectors of the natural basis on L. Here, $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, $|\mathbf{t}| = |\mathbf{n}| = 1$, and, for a line with zero torsion of the plane curve, the Seret-Frenet formulae hold:

$$\mathbf{t}' = k\mathbf{n}, \quad \mathbf{n}' = -k\mathbf{t}, \quad \mathbf{b}' = 0 \tag{1.2}$$

where k is the curvature of the curve L.

In the parameterization adopted (Eq. (1.1)), by virtue of Eqs (1.2), the Lamé parameters

$$H_1 = |\mathbf{R}_1| = |\partial\mathbf{R}/\partial x|, \quad H_2 = |\partial\mathbf{R}/\partial y|, \quad H_3 = |\partial\mathbf{R}/\partial z|$$

will take the form

$$H_1 = 1 - yk, \quad H_2 = H_3 = 1 \tag{1.3}$$

and the tensile strains ε_1, ε_2 and ε_3 and shear strains γ_{12}, γ_{13} and γ_{23} in the complete quadratic approximation, the use of which does not produce "false" bifurcation solutions, are expressed in terms of the displacement components U_1, U_2 and U_3 by the formulae[2,6]

$$\varepsilon_1 = E_{11} + (E_{12}^2 + E_{13}^2)/2, \quad \gamma_{12} = E_{12}(1 + E_{22}) + E_{21}(1 + E_{11}) + E_{13}E_{23}, \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.4}$$

$$E_{11} = \frac{1}{H_1}\left(\frac{\partial U_1}{\partial x^1} + \frac{\partial H_1}{\partial x^2}U_2 + \frac{\partial H_1}{\partial x^3}U_3\right), \quad E_{12} = \frac{1}{H_1}\left(\frac{\partial U_2}{\partial x^1} - \frac{1}{H_2}\frac{\partial H_1}{\partial x^2}U_1\right)$$

$$E_{13} = \frac{1}{H_1}\left(\frac{\partial U_3}{\partial x^1} - \frac{\partial H_1}{\partial x^3}U_1\right), \quad \overrightarrow{\underleftarrow{1, 2, 3}}, \quad \frac{\partial H_1}{\partial x^3} = 0, \quad \frac{\partial H_1}{\partial x^2} = -k$$

$$x^1 = x, \quad x^2 = y, \quad x^3 = z \tag{1.5}$$

Below, we will assume that the rod is thin, and that its curvature k and cross-sectional dimensions satisfy the condition $|yk| \sim \varepsilon$, where $\varepsilon \ll 1$. By virtue of this condition, in formula (1.5), with an accuracy of $1 + \varepsilon \approx 1$, we can assume that $H_1 \approx 1$, and for the displacement vector $\mathbf{U} = U_1\mathbf{t} + U_2\mathbf{n} + U_3\mathbf{b}$ we will adopt the representation

$$\begin{aligned}\mathbf{U} &= \mathbf{u} + \boldsymbol{\varphi} \times (y\mathbf{n} + z\mathbf{b}) + y\gamma_2\mathbf{n} + z\gamma_3\mathbf{b} = (u + z\psi - y\chi)\mathbf{t} + \\ &+ (\upsilon - z\varphi + y\gamma_2)\mathbf{n} + (w + y\varphi + z\gamma_3)\mathbf{b}\end{aligned} \tag{1.6}$$

in which the one-dimensional functions u, υ and w of argument x are components of the displacement vector $\mathbf{u}$ of points of the centreline L, and ψ, χ and φ are components of the vector of revolutions $\boldsymbol{\varphi}$ about the unit vectors $\mathbf{n}$, $\mathbf{b}$ and $\mathbf{t}$, while the functions γ_2 and γ_3 describe the transverse deformations of the rod.

If expressions (1.6) for U_1, U_2 and U_3 are introduced into formulae (1.5), then, within the approximation $H_1 \approx 1$, apart from terms containing y and z, for determining $E_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3$) we can obtain the approximate formulae

$$\begin{aligned}&E_{11} = u' - k\upsilon - y(\chi' + k\gamma_2) + z(\psi' + k\varphi), \quad E_{12} = \upsilon' + ku - z(\varphi' - k\psi) + y(\gamma_2' - k\chi)\\ &E_{13} = w' + y\varphi' + z\gamma_3', \quad E_{21} = -\chi, \quad E_{22} = \gamma_2, \quad E_{23} = \varphi, \quad E_{31} = \psi, \quad E_{32} = -\varphi,\\ &E_{33} = \gamma_3\end{aligned} \tag{1.7}$$

When these are substituted into formulae (1.4), with the same degree of accuracy, for ε_1, we will obtain the approximate expression

$$\varepsilon_1 = \varepsilon_1^0 + z\chi_{12} - y\chi_{13} \tag{1.8}$$

where

$$\begin{aligned}\varepsilon_1^0 &= u' - k\upsilon + (\upsilon' + ku)^2/2 + w'^2/2\\ \chi_{12} &= \psi' + k\varphi - (\varphi' - k\psi)(\upsilon' + ku) + w'\gamma_3'\\ \chi_{13} &= \chi' + k\gamma_2 - w'\varphi' - (\gamma_2' - k\chi)(\upsilon' + ku)\end{aligned} \tag{1.9}$$

for ε_2 and ε_3 we will obtain the "accurate" expressions

$$\varepsilon_2 = \gamma_2 + (\chi^2 + \varphi^2)/2, \quad \varepsilon_3 = \gamma_3 + (\psi^2 + \varphi^2)/2 \tag{1.10}$$

and for the shear deformations γ_{12} and γ_{13} we will obtain unsimplified relations of the form

$$\begin{aligned}
\gamma_{12} &= (\upsilon' + ku)(1 + \gamma_2) - \chi(1 + u' - k\upsilon) + \varphi w' - \\
&- z[(1 + \gamma_2)(\varphi' - k\psi) + \chi(\psi' + k\varphi) - \varphi\gamma_3'] + \\
&+ y[(\gamma_2' - k\chi)(1 + \gamma_2) + \chi(\chi' + k\gamma_2) + \varphi\varphi'] \\
\gamma_{13} &= w'(1 + \gamma_3) + \psi(1 + u' - k\upsilon) - \varphi(\upsilon' + ku) + \\
&+ y[(1 + \gamma_3)\varphi' - \psi(\chi' + k\gamma_2) - \varphi(\gamma_2' - k\chi)] + \\
&+ z[\gamma_3'(1 + \gamma_3) + \psi(\psi' + k\varphi) - \varphi(\varphi' - k\psi)]
\end{aligned} \tag{1.11}$$

In accordance with earlier results,[5] it is acceptable to adopt relation (1.11) in simplified form, ignoring terms with the factor y in the first relation and terms with the factor z in the second. Here, as in expressions (1.8) and (1.10), the principal terms should be retained with the adopted degree of accuracy, which, within the framework of the refined Timoshenko model, enables the tensile–compressive and flexural deformations in the y and z directions and the transverse shear deformations in xy and xz planes to be described by their averaged values over the cross-sections, and also the torsional deformation within the framework of the classical rod model.

2. The geometrically non-linear equations of equilibrium

Introducing the hypothesis $\sigma_{23} = 0$ to describe the deformation of rods, we obtain the following expression for the variation of the deformation potential energy

$$\delta U = \iiint_V (\sigma_{11}\delta\varepsilon_1 + \sigma_{22}\delta\varepsilon_2 + \sigma_{33}\delta\varepsilon_3 + \sigma_{12}\delta\gamma_{12} + \sigma_{13}\delta\gamma_{13})dV \tag{2.1}$$

After using expressions (1.8) to (1.11) and introducing the notation for the internal forces and moments

$$Q_x = \iint_F \sigma_{11} dF, \quad Q_y = \iint_F \sigma_{12} dF, \quad Q_z = \iint_F \sigma_{13} dF$$

$$\begin{aligned}
&M_y = \iint_F \sigma_{11} z dF, \quad M_z = -\iint_F \sigma_{11} y dF, \quad M_{xy} = \iint_F \sigma_{12} z dF \\
&M_{xz} = \iint_F \sigma_{13} y dF, \quad M_x = M_{xz} - M_{xy} = \iint_F (\sigma_{13} y - \sigma_{12} z) dF \\
&T_y = \iint_F \sigma_{22} dF, \quad T_z = \iint_F \sigma_{33} dF, \quad S_{xy} = \iint_F \sigma_{12} y dF, \quad S_{xz} = \iint_F \sigma_{13} z dF
\end{aligned} \tag{2.2}$$

we can reduce expression (2.1) to the form

$$\begin{aligned}
\delta U = \int_{x^-}^{x^+} (&Q_x^* \delta u' + Q_y^* \delta \upsilon' + Q_z^* \delta w' + S_x^* \delta u + S_y^* \delta \upsilon + M_y^* \delta \psi' + M_z^* \delta \chi' + M_x^* \delta \varphi' + \\
&+ N_x^* \delta \varphi + N_y^* \delta \chi + N_z^* \delta \psi + S_{xy}^* \delta \gamma_2' + S_{xz}^* \delta \gamma_3' + T_y^* \delta \gamma_2 + T_z^* \delta \gamma_3) dx
\end{aligned} \tag{2.3}$$

where

$$Q_x^* = Q_x - Q_y\chi + Q_z\psi$$

$$Q_y^* = Q_y(1+\gamma_2) + Q_x(v'+ku) - Q_z\varphi - M_y(\varphi' - k\psi) - M_z(\gamma_2' - k\chi)$$

$$Q_z^* = Q_z(1+\gamma_3) + Q_x w' + Q_y\varphi + M_y\gamma_3' - M_z\varphi'$$

$$S_x^* = k[Q_x(v'+ku) + Q_y(1+\gamma_2) - Q_z\varphi - M_y(\varphi'-k\psi) - M_z(\gamma_2'-k\chi)], \quad S_y^* = -Q_x^* k$$

$$M_y^* = M_y - M_{xy}\chi + S_{xz}\psi, \quad M_z^* = M_z - M_{xz}\psi + S_{xy}\chi$$

$$M_x^* = M_x - M_{xy}\gamma_2 + M_{xz}\gamma_3 - M_y(v'+ku) - M_z w' + (S_{xy}+S_{xy})\varphi$$

$$N_x^* = Q_y w' - Q_z(v'+ku) + (T_y + T_z)\varphi + kM_x\chi +$$

$$+ M_y k - M_{xz}(\gamma_2' - k\chi) + M_{xy}\gamma_3' + (S_{xy}+S_{xz})\varphi$$

$$N_y^* = -Q_y(1+u'-kv) + M_z k(v'+ku) + M_x k\varphi - M_{xy}\psi' + T_y\chi + S_{xy}(\chi'-k)$$

$$N_z^* = Q_z(1+u'-kv) + M_y k(v'+ku) - M_x k\gamma_2 + M_{xy}k - M_{xz}\chi' + T_z\psi + S_{xz}\psi'$$

$$S_{xy}^* = S_{xy}(1+\gamma_2) - M_z(v'+ku) - M_{xz}\varphi, \quad S_{xz}^* = S_{xz}(1+\gamma_3) + M_y w' + M_{xy}\varphi$$

$$T_y^* = T_y + Q_y(v'+ku) + M_z k - M_{xy}\varphi' - M_x k\psi + S_{xy}\gamma_2', \; T_z^* = T_z + Q_z w' + M_{xz}\varphi' + S_{xz}\gamma_3' \quad (2.4)$$

If the rod material is orthotropic, where the orthotropy axes coincide with the x, y, and z axes, then the stress components for linearly elastic behaviour that occur in expressions (2.1) and (2.2) are related to the strain components (1.8), (1.10) and (1.11) by the equations of elasticity

$$\sigma_{11} = g_{11}\varepsilon_1 + g_{12}\varepsilon_2 + g_{13}\varepsilon_3, \quad \overrightarrow{\underleftarrow{1, 2, 3}}; \quad \sigma_{12} = G_{12}\gamma_{12}, \quad \sigma_{13} = G_{13}\gamma_{13} \quad (2.5)$$

in which G_{12} and G_{13} denote the shear moduli, while the elastic characteristics $g_{\alpha\beta} = g_{\beta\alpha}$ ($\alpha, \beta = 1, 2, 3$) are expressed in terms of the elastic moduli E_1, E_2 and E_3 and Poisson's ratios $\nu_{\alpha\beta}$ by the well-known relations (see, for example, Ref. 7).

Below we will assume that, in each cross-section of the rod x = const, the y and z axes are the principal central axes of inertia. Then, from expressions (1.8), (1.10), (1.11), (2.5) and (2.2), we obtain the physical relations

$$Q_x = F(g_{11}\overset{0}{\varepsilon}_1 + g_{12}\varepsilon_2 + g_{13}\varepsilon_3)$$

$$T_y = F(g_{12}\overset{0}{\varepsilon}_1 + g_{22}\varepsilon_2 + g_{23}\varepsilon_3), \quad T_z = F(g_{13}\overset{0}{\varepsilon}_1 + g_{23}\varepsilon_2 + g_{33}\varepsilon_3)$$

$$M_y = g_{11}J_y\chi_{12}, \quad M_z = g_{11}J_z\chi_{13}$$

$$Q_y = G_{12}F[(v'+ku)(1+\gamma_2) - \chi(1+u'-kv) + \varphi w']$$

$$Q_z = G_{13}F[w'(1+\gamma_3) + \psi(1+u'-kv) - \varphi(v'+ku)]$$

$$M_{xy} = -G_{12}J_y[(1+\gamma_2)(\varphi'-k\psi) + \chi(\psi'+k\varphi) - \varphi\gamma_3']$$

$$M_{xz} = G_{13}J_z[(1+\gamma_3)\varphi' - \psi(\chi'+k\gamma_2) - \varphi(\gamma_2'-k\chi)]$$

$$S_{xy} = G_{12}J_z[(1+\gamma_2)(\gamma_2'-k\chi) + \chi(\chi'+k\gamma_2) + \varphi\varphi']$$

$$S_{xz} = G_{13}J_y[(1+\gamma_3)\gamma_3' + \psi(\psi'+k\varphi) + \varphi(\varphi'-k\psi)] \quad (2.6)$$

where J_y and J_z are the principal moments of inertia of the transverse cross-section.

Suppose the contour of each cross-section of the rod x = const is specified by the parametric equations $y_l = y_l(l)$ and $z_l = z_l(l)$, where l is the length of the arc along the directrix, and all external surface forces acting on the rod at points of its lateral surface S are represented by the expansion

$$\mathbf{p}_s = p_{st}\mathbf{t} + p_{sn}\mathbf{n} + p_{sb}\mathbf{b}$$

The variation of the work of these forces on possible displacements

$$\delta\mathbf{U}_s = \delta(u + z_l\psi - y_l\chi)\mathbf{t} + \delta(\upsilon - z_l\varphi + y_l\gamma_2)\mathbf{n} + \delta(w + y_l\varphi + z_l\gamma_3)\mathbf{b}$$

and also of the volume forces

$$\mathbf{F}(x) = F_1(x)\mathbf{t} + F_2(x)\mathbf{n} + F_3(x)\mathbf{b}$$

on possible displacements $\delta\mathbf{U}$ will be equal to

$$\delta A_1 = \iiint_V \mathbf{F}\delta\mathbf{U}dV + \int_{x^-}^{x^+}\oint_L \mathbf{p}_s\delta\mathbf{U}_s dl dx = \int_{x^-}^{x^+}(X_1\delta u + X_2\delta\upsilon + X_3\delta w +$$
$$+ m_z\delta\chi + m_y\delta\psi + m_x\delta\varphi + t_2\delta\gamma_2 + t_3\delta\gamma_3)dx \tag{2.7}$$

where, by virtue of the fact that the y and z axes in each cross-section are the principal central axes of inertia for the external forces and moments introduced into consideration, reduced to the rod centreline, the following formulae hold

$$X_1 = FF_1 + \oint_L p_{st}dl, \quad X_2 = FF_2 + \oint_L p_{sn}dl, \quad X_3 = FF_3 + \oint_L p_{sb}dl$$

$$m_z = \oint_L p_{st}y_l dl, \quad m_y = \int_L p_{st}z_l dl, \quad m_x = \int_L (p_{sb}y_l - p_{sn}z_l)dl$$

$$t_2 = \int_L p_{sn}y_l dl, \quad t_3 = \int_L p_{sb}z_l dl \tag{2.8}$$

Below, the external surface forces $\mathbf{p}_s$ introduced into consideration will be regarded as reduced to four linear force vectors $\mathbf{P}^+$, $\mathbf{P}^-$, $\mathbf{\Phi}^+$ and $\mathbf{\Phi}^-$ applied at points on the lines with coordinates $(x, y^+, 0)$, $(x, y^-, 0)$, $(x, 0, z^+)$ and $(x, 0, z^-)$. The displacement vectors of points on these lines will be

$$\mathbf{U}_s(x, y^\pm, 0) = (u - y^\pm\chi)\mathbf{t} + (\upsilon + y^\pm\gamma_2)\mathbf{n} + (w + y^\pm\varphi)\mathbf{b}$$
$$\mathbf{U}_s(x, 0, z^\pm) = (u + z^\pm\psi)\mathbf{t} + (\upsilon - z^\pm\varphi)\mathbf{n} + (w + z^\pm\gamma_3)\mathbf{b} \tag{2.9}$$

If the force vectors $\mathbf{P}^\pm$ and $\mathbf{F}^\pm$ are specified by the expansions

$$\mathbf{P}^\pm = P_1^\pm\mathbf{t} + P_2^\pm\mathbf{n} + P_3^\pm\mathbf{b}, \quad \mathbf{\Phi}^\pm = \Phi_1^\pm\mathbf{t} + \Phi_2^\pm\mathbf{n} + \Phi_3^\pm\mathbf{b} \tag{2.10}$$

and, during deformation, they maintain their directions, then, by using expressions (2.9), formulae (2.8) will take the form

$$X_j = F_jF + P_j^+ + P_j^- + \Phi_j^+ + \Phi_j^-, \quad j = 1, 2, 3$$
$$m_z = -P_1^+y^+ - P_1^-y^-, \quad m_y = \Phi_1^+z^+ + \Phi_1^-z^-$$
$$m_x = P_3^+y^+ + P_3^-y^- - \Phi_2^+z^+ - \Phi_2^-z^-$$
$$t_2 = P_2^+y^+ + P_2^-y^-, \quad t_3 = \Phi_3^+z^+ + \Phi_3^-z^- \tag{2.11}$$

We will use $\mathbf{p}_1$ to denote the vectors of the specified surface forces applied at points of the end cross-sections $x = x^-$ and $x = x^+$. If, during deformation of the rod, these vectors retain their directions and are specified by the expansions

$$\mathbf{p}_1 = p_{11}\mathbf{t} + p_{12}\mathbf{n} + p_{13}\mathbf{b} \tag{2.12}$$

then the work completed by them on variations of the corresponding displacements will be equal to

$$\delta A_2 = [Q_x^s\delta u + Q_y^s\delta\upsilon + Q_z^s\delta w + M_y^s\delta\psi + M_z^s\delta\chi + M_x^s\delta\varphi + S_{xy}^s\delta\gamma_2 + S_{xz}^s\delta\gamma_3]\Big|_{x = x^-}^{x = x^+} \tag{2.13}$$

where

$$Q_x^s = \iint_F p_{11}dF, \quad Q_y^s = \iint_F p_{12}dF, \quad Q_z^s = \iint_F p_{13}dF$$

$$M_y^s = \iint_F p_{11}zdF, \quad M_z^s = -\iint_F p_{11}ydF, \quad M_{xy}^s = \iint_F p_{12}zdF, \quad M_{xz}^s = \iint_F p_{13}ydF$$

$$M_x^s = M_{xz}^s - M_{xy}^s = \iint_F (p_{13}y - p_{12}z)dF, \quad S_{xy}^s = \iint_F p_{12}ydF, \quad S_{xz}^s = \iint_F p_{13}zdF \tag{2.14}$$

Now, introducing expressions (2.3), (2.9) and (2.13) into the variational equation of the principle of virtual displacements $\delta U - \delta A_1 - \delta A_2 = 0$, we obtain the equation

$$\begin{aligned}
&[(Q_x^* - Q_x^s)\delta u + (Q_y^* - Q_y^s)\delta \upsilon + (Q_z^* - Q_z^s)\delta w + \\
&+ (M_y^* - M_y^s)\delta\psi + (M_z^* - M_z^s)\delta\chi + (M_x^* - M_x^s)\delta\varphi + \\
&+ (S_{xy}^* - S_{xy}^s)\delta\gamma_2 + (S_{xz}^* - S_{xz}^s)\delta\gamma_3]\Big|_{x=x^-}^{x=x^+} - \\
&- \int_0^a (f_1^*\delta u + f_2^*\delta\upsilon + f_3^*\delta w + f_4^*\delta\psi + f_5^*\delta\chi + f_6^*\delta\varphi + f_7^*\delta\gamma_2 + f_8^*\delta\gamma_3)dx = 0
\end{aligned} \tag{2.15}$$

from which we obtain a system of eight ordinary differential equations of equilibrium

$$\begin{aligned}
&f_1^* = Q_x^{*\prime} - S_x^* + X_1 = 0, \quad f_2^* = Q_y^{*\prime} - S_y^* + X_2 = 0 \\
&f_3^* = Q_z^{*\prime} + X_3 = 0, \quad f_4^* = M_y^{*\prime} - N_z^* + m_y = 0 \\
&f_5^* = M_z^{*\prime} - N_y^* + m_z = 0, \quad f_6^* = M_x^{*\prime} - N_x^* + m_x = 0 \\
&f_7^* = S_{xy}^{*\prime} - T_y^* + t_2 = 0, \quad f_8^* = S_{xz}^{*\prime} - T_z^* + t_3 = 0
\end{aligned} \tag{2.16}$$

relative to the undeformed axes, and the corresponding static boundary conditions

$$\begin{aligned}
&Q_x^* = Q_x^s \text{ when } \delta u \neq 0, \quad Q_y^* = Q_y^s \text{ when } \delta\upsilon \neq 0, \quad Q_z^* = Q_z^s \text{ when } \delta w \neq 0 \\
&M_y^* = M_y^s \text{ when } \delta\psi \neq 0, \quad M_z^* = M_z^s \text{ when } \delta\chi \neq 0, \quad M_x^* = M_x^s \text{ when } \delta\varphi \neq 0 \\
&S_{xy}^* = S_{xy}^s \text{ when } \delta\gamma_2 \neq 0, \quad S_{xz}^* = S_{xz}^s \text{ when } \delta\gamma_3 \neq 0
\end{aligned} \tag{2.17}$$

If the directions of action of the external forces change during the deformation of the rod, then it must be ascertained how their orientation will change in relation to the vectors **t**, **n** and **b**. Besides these vectors, the following basis vectors are also associated with the space of the rod

$$\begin{aligned}
&\mathbf{R}_1^* = (1 + E_{11})\mathbf{t} + E_{12}\mathbf{n} + E_{13}\mathbf{b}, \quad \mathbf{R}_2^* = -\chi\mathbf{t} + (1 + \gamma_2)\mathbf{n} + \varphi\mathbf{b} \\
&\mathbf{R}_3^* = \psi\mathbf{t} - \varphi\mathbf{n} + (1 + \gamma_3)\mathbf{b}
\end{aligned} \tag{2.18}$$

Of these, the vectors $\mathbf{R}_2^*$ and $\mathbf{R}_3^*$ retain their directions at all points of the cross-section, while the vector $\mathbf{R}_1^*$ at points of the deformed centre-line L_* will be equal to

$$\mathbf{R}_1^*(x, 0, 0) = (1 + u' - k\upsilon)\mathbf{t} + (\upsilon' + ku)\mathbf{n} + w'\mathbf{b} \tag{2.19}$$

In accordance with Eq. (2.19) for the unit vector $\mathbf{t}^*$ tangential to line L_*, we have the expression

$$\mathbf{t}^* = \zeta_1[(1 + u' - k\upsilon)\mathbf{t} + (\upsilon' + ku)\mathbf{n} + w'\mathbf{b}] \tag{2.20}$$

where

$$\zeta_1 = 1/(1 + 2\tilde{\varepsilon}_{11})^{1/2}, \quad 2\tilde{\varepsilon}_{11} = 2\varepsilon_1 + (u' - k\upsilon)^2$$

Using expression (2.20), from well-known formulae of differential geometry we can obtain expressions for two other unit vectors $\mathbf{n}^*$ and $\mathbf{b}^*$ of the accompanying triangle of the deformed curve L_*. Under small strains, the following unit vectors are practically identical to them in direction

$$\mathbf{r}_2^* = \mathbf{R}_2^*/|\mathbf{R}_2^*| = \zeta_2[-\chi\mathbf{t} + (1+\gamma_2)\mathbf{n} + \varphi\mathbf{b}]$$
$$\mathbf{r}_3^* = \mathbf{R}_3^*/|\mathbf{R}_3^*| = \zeta_3[\psi\mathbf{t} + \varphi\mathbf{n} + (1+g_3)\mathbf{b}] \tag{2.21}$$

where

$$\zeta_2 = 1/(1+2\tilde{\varepsilon}_2)^{1/2}, \quad \zeta_3 = 1/(1+2\tilde{\varepsilon}_3)^{1/2}, \quad \tilde{\varepsilon}_2 = \varepsilon_2 + \gamma_2^2/2, \quad \tilde{\varepsilon}_3 = \varepsilon_3 + \gamma_3^2/2$$

In the case of small tensile strains ε_1, ε_2 and ε_3, in formulae (2.20) and (2.21) it can be assumed that $\zeta_1 \approx 1$, $\zeta_2 \approx 1$ and $\zeta_3 \approx 1$. Here, with the aim of constructing a consistent version of rod theory in a non-linear quadratic approximation in accordance with results obtained earlier,[5,6] simplified expressions must be adopted for the vectors $\mathbf{t}^*$, $\mathbf{r}_2^*$ and $\mathbf{r}_3^*$:

$$\mathbf{t}^* = \mathbf{t} + (v' + ku)\mathbf{n} + w'\mathbf{b} \tag{2.22}$$

$$\mathbf{r}_2^* = -\chi\mathbf{t} + \mathbf{n} + \varphi\mathbf{b}, \quad \mathbf{r}_3^* = \psi\mathbf{t} - \varphi\mathbf{n} + \mathbf{b} \tag{2.23}$$

Then, using these expressions, we obtain the equations

$$\mathbf{t}^*\mathbf{r}_2^* = -\chi + v' + ku + w'\varphi = 0, \quad \mathbf{t}^*\mathbf{r}_3^* = \psi - \varphi(v' + ku) + w' = 0$$

from which we then find the relations

$$\chi = v' + ku + w'\varphi, \quad \psi = -w' + (v' + ku)\varphi \tag{2.24}$$

Introducing these relations into expressions (2.23), we obtain the two unit vectors

$$\mathbf{n}^* = -(v' + ku + w'\varphi)\mathbf{t} + \mathbf{n} + \varphi\mathbf{b}, \quad \mathbf{b}^* = [(v' + ku)\varphi - w']\mathbf{t} - \varphi\mathbf{n} + \mathbf{b} \tag{2.25}$$

which, while remaining orthogonal to the vector $\mathbf{t}^*$ during the deformation of the rod, with the adopted degree of accuracy, can be considered, together with vector (2.22), to be identical to the vectors of the accompanying triangle of the deformed curve L_*.

The vectors of the surface forces $\mathbf{p}_1$ applied in the end cross-sections $x = x^-$ and $x = x^+$ will now be represented in the form of the expansion

$$\mathbf{p}_1 = \tilde{p}_{11}\mathbf{t}^* + \tilde{p}_{12}\mathbf{n}^* + \tilde{p}_{13}\mathbf{b}^* \tag{2.26}$$

Introducing expressions (2.22) and (2.25), we obtain a representation in the form of Eq. (2.12), in which the components p_{11}, p_{12} and p_{13} are not specified but are expressed in terms of the specified components p_{11}, p_{12} and p_{13} by the formulae

$$p_{11} = \tilde{p}_{11} - \tilde{p}_{12}(v' + ku + w'\varphi) + \tilde{p}_{13}[(v' + ku)\varphi - w']$$
$$p_{12} = \tilde{p}_{11}(v' + ku) + \tilde{p}_{12} - \tilde{p}_{13}\varphi, \quad p_{13} = \tilde{p}_{11}w' + \tilde{p}_{12}\varphi + \tilde{p}_{13} \tag{2.27}$$

If, by analogy with system (2.14), we introduce the notation

$$\tilde{Q}_x^s = \iint_F \tilde{p}_{11}dF, \quad \ldots, \quad \tilde{M}_x^s = \tilde{M}_{xz}^s - \tilde{M}_{xy}^s = \iint_F (\tilde{p}_{13}y - \tilde{p}_{12}z) \tag{2.28}$$

then, by substituting relations (2.27) into formulae (2.14), with an accuracy $\iint_F \tilde{p}_{12}ydF = \iint_F p_{13}zdF = 0$ we can obtain the relations

$$Q_x^s = \tilde{Q}_x^s - \tilde{Q}_y^s(v' + ku + w'\varphi) + Q_z^s[(v' + ku)\varphi - w']$$

$$Q_y^s = \tilde{Q}_x^s(v' + ku) + \tilde{Q}_y^s - Q_z^s\varphi, \quad Q_z^s = \tilde{Q}_x^s w' + \tilde{Q}_y^s\varphi + Q_z^s$$

$$M_y^s = \tilde{M}_y^s - \tilde{M}_{xy}^s(v' + ku + w'\varphi) + \tilde{S}_{xz}[(v' + ku)\varphi - w']$$

$$M_z^s = \tilde{M}_z^s - \tilde{M}_{xz}^s[(v' + ku)\varphi - w'] + \tilde{S}_{xy}^s(v' + ku + w'\varphi)$$

$$M_{xy}^s = \tilde{M}_{xy}^s + \tilde{M}_y^s(v' + ku) - \tilde{S}_{xz}^s\varphi$$

$$M_{xz}^s = \tilde{M}_{xz}^s - \tilde{M}_z^s w' + S_{xy}^s\varphi, \quad M_x^s = -\tilde{M}_z^s w' - \tilde{M}_y^s(v' + ku) + \tilde{M}_x^s + (\tilde{S}_{xy}^s + \tilde{S}_{xz}^s)\varphi$$

$$S_{xy}^s = \tilde{S}_{xy}^s - \tilde{M}_z^s(v' + ku) - M_{xz}^s\varphi, \quad S_{xz} = \tilde{S}_{xz}^s + \tilde{M}_y^s w' + \tilde{M}_{xy}^s\varphi \tag{2.29}$$

which can be used to formulate the boundary conditions in the cross-sections $x = x^-$ and $x = x^+$, depending on the clamping conditions and the form of end loading expressed by Eq. (2.26). For example, when $\tilde{p}_{11} = \text{const}$ and $\tilde{p}_{12} = \tilde{p}_{13} = 0$, from formulae (2.28) we will have

$$\tilde{Q}_x^s = \tilde{p}_{11}F, \quad \tilde{Q}_y^s = \tilde{Q}_z^s = \tilde{M}_y^s = \tilde{M}_z^s = \tilde{M}_{xy}^s = \tilde{M}_{xz}^s = \tilde{M}_x^s = S_{sy}^s = S_{xz}^s = 0 \tag{2.30}$$

Therefore, if kinematic clamping conditions are imposed in the cross-sections $x = x^-$ and $x = x^+$

$$\upsilon = w = 0 \tag{2.31}$$

and $\delta\psi \neq 0$, $\delta\chi \neq 0$, $\delta\gamma_2 \neq 0$ and $\delta\gamma_3 = 0$, then, on the basis of conditions (2.17), taking Eqs (2.31) into account, and using relations (2.29), the following static boundary conditions will be formulated

$$Q_x^* = \tilde{p}_{11}F, \quad M_y^* = M_z^* = M_x^* = S_{sy}^* = S_{xz}^* = 0$$

which correspond to an end compression of the rod that 'follows' the direction of vector $\mathbf{t}^*$.

Along with Eq. (2.26), for $\mathbf{p}_1$ it is advisable also to consider the equation

$$\mathbf{p}_1 = \tilde{\tilde{p}}_{11}\mathbf{t}^* + \tilde{\tilde{p}}_{12}\mathbf{r}_2^* + \tilde{\tilde{p}}_{13}\mathbf{r}_3^* \tag{2.32}$$

the use of which, together with Eq. (2.12) and relations (2.22) and (2.23), leads to the relations

$$\begin{aligned} &p_{11} = \tilde{\tilde{p}}_{11} - \tilde{\tilde{p}}_{12}\chi + \tilde{\tilde{p}}_{13}\psi, \quad p_{12} = \tilde{\tilde{p}}_{11}(\upsilon' + ku) + \tilde{\tilde{p}}_{12} - \tilde{\tilde{p}}_{13}\varphi \\ &p_{13} = \tilde{\tilde{p}}_{11}w' + \tilde{\tilde{p}}_{12}\varphi + \tilde{\tilde{p}}_{13} \end{aligned} \tag{2.33}$$

By introducing these relations into formulae (2.14), we will obtain

$$\begin{aligned} &Q_x^s = \tilde{\tilde{Q}}_x^s - \tilde{\tilde{Q}}_y^s\chi + \tilde{\tilde{Q}}_z^s\psi, \quad Q_y^s = \tilde{\tilde{Q}}_x^s(\upsilon' + ku) + \tilde{\tilde{Q}}_y^s + \tilde{\tilde{Q}}_z^s\varphi \\ &Q_z^s = \tilde{\tilde{Q}}_x^s w' + \tilde{\tilde{Q}}_y^s\varphi + \tilde{\tilde{Q}}_z^s, \quad M_y^s = \tilde{\tilde{M}}_y^s - \tilde{\tilde{M}}_{xy}^s\chi + \tilde{\tilde{S}}_{xz}^s\psi \\ &M_z^s = \tilde{\tilde{M}}_z^s - \tilde{\tilde{M}}_{xz}^s\psi + \tilde{\tilde{S}}_{xy}^s\chi, \quad M_{xy}^s = \tilde{\tilde{M}}_y^s(\upsilon' + ku) + \tilde{\tilde{M}}_{xy}^s - \tilde{\tilde{S}}_{xz}^s\varphi \\ &M_{xz}^s = -\tilde{\tilde{M}}_z^s w' + \tilde{\tilde{M}}_{xz}^s + \tilde{\tilde{S}}_{xy}^s\varphi, \quad M_x^s = -\tilde{\tilde{M}}_z^s w' - \tilde{\tilde{M}}_y^s(\upsilon' + ku) + \\ &+ \tilde{\tilde{M}}_x^s + (\tilde{\tilde{S}}_{xy}^s + \tilde{\tilde{S}}_{xz}^s)\varphi, \quad S_{xy}^s = -\tilde{\tilde{M}}_z^s(\upsilon' + ku) + \tilde{\tilde{S}}_{xy}^s - \tilde{\tilde{M}}_{xz}^s\varphi \\ &S_{xz}^s = \tilde{\tilde{M}}_y^s w' + \tilde{\tilde{M}}_{xy}^s\varphi + \tilde{\tilde{S}}_{xz}^s \end{aligned} \tag{2.34}$$

where $\tilde{\tilde{M}}_y^s, \ldots, \tilde{\tilde{S}}_{xz}^s$ denote end forces and moments similar to the internal forces and moments described by formulae (2.14) but calculated in terms of the components $\tilde{\tilde{p}}_\alpha$.

However, if instead of Eq. (2.32) for $\mathbf{p}_1$ we adopt the equation

$$\mathbf{p}_1 = \tilde{\tilde{p}}_{11}\mathbf{l}_1^* + \tilde{\tilde{p}}_{12}\mathbf{l}_2^* + \tilde{\tilde{p}}_{13}\mathbf{l}_3^* \tag{2.35}$$

where

$$\mathbf{l}_\alpha^* = \mathbf{R}_\alpha^*/|\mathbf{R}_\alpha^*|, \quad \mathbf{R}_\alpha^* = \partial(\mathbf{R} + \mathbf{U})/\partial x^\alpha, \quad \alpha = 1, 2, 3$$

then the components $p_{1\alpha}$ and $\tilde{\tilde{p}}_{1\alpha}$ in Eqs (2.12) and (2.35) will be related by the equations

$$\begin{aligned} &p_{11} = \tilde{\tilde{p}}_{11} + \tilde{\tilde{p}}_{12}E_{21} + \tilde{\tilde{p}}_{13}E_{31}, \quad p_{12} = \tilde{\tilde{p}}_{11}E_{12} + \tilde{\tilde{p}}_{12}(1 + E_{22}) + \tilde{\tilde{p}}_{13}E_{32} \\ &p_{13} = \tilde{\tilde{p}}_{11}E_{13} + \tilde{\tilde{p}}_{12}E_{23} + \tilde{\tilde{p}}_{13}(1 + E_{33}) \end{aligned}$$

which, using expressions (1.7), are transformed into

$$\begin{aligned} &p_{11} = \tilde{\tilde{p}}_{11} - \tilde{\tilde{p}}_{12}\chi - \tilde{\tilde{p}}_{13}\psi, \quad p_{12} = \tilde{\tilde{p}}_{11}[\upsilon' + ku - z(\varphi' - k\psi) + y(\gamma_2' - k\chi)] + \\ &+ \tilde{\tilde{p}}_{12}(1 + \gamma_2) - \tilde{\tilde{p}}_{13}\varphi, \quad p_{13} = \tilde{\tilde{p}}_{11}(w' + y\varphi' + z\gamma_3') + \tilde{\tilde{p}}_{12}\varphi + \tilde{\tilde{p}}_{13}(1 + \gamma_3) \end{aligned} \tag{2.36}$$

and, by substitution into formulae (2.14), are reduced to the one-dimensional relations

$$
\begin{aligned}
Q_x^s &= \tilde{\tilde{Q}}_x^s - \tilde{\tilde{Q}}_y^s\chi + \tilde{\tilde{Q}}_z^s\psi \\
Q_y^s &= \tilde{\tilde{Q}}_x^s(v' + ku) - \tilde{\tilde{M}}_y^s(\varphi' - k\psi) - \tilde{\tilde{M}}_z^s(\gamma_2' - k\chi) + \tilde{\tilde{Q}}_y^s(1+\gamma_2) - \tilde{\tilde{Q}}_z^s\varphi \\
Q_z^s &= \tilde{\tilde{Q}}_x^s w' - \tilde{\tilde{M}}_z^s\varphi' + \tilde{\tilde{M}}_y^s\gamma_3' + \tilde{\tilde{Q}}_y^s\varphi + \tilde{\tilde{Q}}_z^s(1+\gamma_3) \\
M_y^s &= \tilde{\tilde{M}}_y^s - \tilde{\tilde{M}}_{xy}^s\chi + \tilde{\tilde{S}}_{xz}^s\psi, \quad M_z^s = \tilde{\tilde{M}}_z^s - \tilde{\tilde{M}}_{xz}^s\psi + \tilde{\tilde{S}}_{xy}^s\chi \\
M_x^s &= -\tilde{\tilde{M}}_y^s(v' + ku) - \tilde{\tilde{M}}_z^s w' + \tilde{\tilde{M}}_x^s + \tilde{\tilde{M}}_{xz}^s\gamma_3 - \tilde{\tilde{M}}_{xy}^s\gamma_2 + (\tilde{\tilde{S}}_{xy}^s + \tilde{\tilde{S}}_{xz}^s)\varphi \\
M_{xy}^s &= \tilde{\tilde{M}}_y^s(v' + ku) + \tilde{\tilde{M}}_{xy}^s(1+\gamma_2) - \tilde{\tilde{S}}_{xz}^s\varphi \\
M_{xz}^s &= -\tilde{\tilde{M}}_z^s w' + \tilde{\tilde{M}}_{xz}^s(1+\gamma_3) + \tilde{\tilde{S}}_{xy}^s\varphi \\
S_{xy}^s &= \tilde{\tilde{S}}_{xy}^s(1+\gamma_2) - \tilde{\tilde{M}}_z^s(v' + ku) - \tilde{\tilde{M}}_{xz}^s\varphi \\
S_{xz}^s &= S_{xz}(1+\gamma_3) + \tilde{\tilde{M}}_y^s w' + \tilde{\tilde{M}}_{xy}^s\varphi
\end{aligned}
\tag{2.37}
$$

Note that, unlike to approximate relations (2.34), the established relations (2.37) have the same degree of accuracy as one-dimensional relations (2.4). They are simplified considerably if the surface forces $\tilde{\tilde{p}}_{11}$, $\tilde{\tilde{p}}_{12}$ and $\tilde{\tilde{p}}_{13}$ are calculated by means of the formulae of classical rod theory

$$
\tilde{\tilde{p}}_{11} = \frac{\tilde{\tilde{Q}}_x^s}{F} - y\frac{\tilde{\tilde{M}}_z^s}{J_z} + z\frac{\tilde{\tilde{M}}_y^s}{J_y}, \quad \tilde{\tilde{p}}_{12} = \frac{\tilde{\tilde{Q}}_y^s}{F} - z\frac{\tilde{\tilde{M}}_x^s}{J_p}, \quad \tilde{\tilde{p}}_{13} = \frac{\tilde{\tilde{Q}}_z^s}{F} + y\frac{\tilde{\tilde{M}}_x^s}{J_p}
\tag{2.38}
$$

in accordance with which $\tilde{\tilde{S}}_{xy}^s = \tilde{\tilde{S}}_{xz}^s = 0$ and only six forces and moments $\tilde{\tilde{Q}}_x^s$, $\tilde{\tilde{Q}}_y^s$, $\tilde{\tilde{Q}}_z^s$, $\tilde{\tilde{M}}_x^s$, $\tilde{\tilde{M}}_y^s$ and $\tilde{\tilde{M}}_z^s$, which are statically equivalent to the surface forces, need be considered specified. Here, for the moments $\tilde{\tilde{M}}_{xy}^s$ and $\tilde{\tilde{M}}_{xz}^s$, the following formulae hold

$$
\tilde{\tilde{M}}_{xy}^s = -\frac{J_y}{J_p}\tilde{\tilde{M}}_x^s, \quad \tilde{\tilde{M}}_{xz}^s = \frac{J_z}{J_p}\tilde{\tilde{M}}_x^s
\tag{2.39}
$$

the substitution of which into the formula $\tilde{\tilde{M}}_x^s = \tilde{\tilde{M}}_{xz}^s - \tilde{\tilde{M}}_{xy}^s$ leads to an identity.

We will now assume that the components of the vectors $\mathbf{P}^{\pm}$ and $\mathbf{\Phi}^{\pm}$ also "follow" the directions of the vectors $\mathbf{t}^*$, $\mathbf{n}^*$ and $\mathbf{b}^*$ defined by formulae (2.22) and (2.23), i.e., we have

$$
\mathbf{P}^{\pm} = \tilde{P}_1^{\pm}\mathbf{t}^* + \tilde{P}_2^{\pm}\mathbf{n}^* + \tilde{P}_3^{\pm}\mathbf{b}^*, \quad \mathbf{\Phi}^{\pm} = \tilde{\Phi}_1^{\pm}\mathbf{t}^* + \tilde{\Phi}_2^{\pm}\mathbf{n}^* + \tilde{\Phi}_3^{\pm}\mathbf{b}^*
\tag{2.40}
$$

Introducing expressions (2.22) and (2.23) here, we obtain equations of the form (2.10), in which the components $P_j^{\pm}$ and $\Phi_j^{\pm}$ will be expressed in terms of the specified components $\tilde{P}_j^{\pm}$ and $\tilde{\Phi}_j^{\pm}$ ($j = 1, 2, 3$) by the formulae

$$
\begin{aligned}
P_1^{\pm} &= \tilde{P}_1^{\pm} - \tilde{P}_2^{\pm}(v' + ku + w'\varphi) + \tilde{P}_3^{\pm}[(v' + ku)\varphi - w'] \\
P_2^{\pm} &= \tilde{P}_1^{\pm}(v' + ku) + \tilde{P}_2^{\pm} - \tilde{P}_3^{\pm}\varphi, \quad P_3^{\pm} = \tilde{P}_1^{\pm}w' + \tilde{P}_2^{\pm}\varphi + \tilde{P}_3^{\pm} \\
\Phi_1^{\pm} &= \tilde{\Phi}_1^{\pm} - \tilde{\Phi}_2^{\pm}(v' + ku + w'\varphi) + \tilde{\Phi}_3^{\pm}[(v' + ku)\varphi - w'] \\
\Phi_2^{\pm} &= \tilde{\Phi}_1^{\pm}(v' + ku) + \tilde{\Phi}_2^{\pm} - \Phi_3^{\pm}\varphi, \quad \Phi_3^{\pm} = \tilde{\Phi}_1^{\pm}w' + \tilde{\Phi}_2^{\pm}\varphi + \tilde{\Phi}_3^{\pm}
\end{aligned}
\tag{2.41}
$$

Substituting these formulae into Eqs (2.17), the forces and moments of the external forces in the equilibrium equations (2.16) are determined.

Along with the derived version of relations (2.41), it is advisable also to set up similar relations that correspond to loading of the rod with forces that "follow" the directions of the basis vectors $\mathbf{t}^*$, $\mathbf{r}_2^*$ and $\mathbf{r}_3^*$ of the deformed state. In this case, the components of the vectors

$\mathbf{P}^{\pm}$ and $\mathbf{\Phi}^{\pm}$ in the equations

$$\mathbf{P}^{\pm} = \tilde{\tilde{P}}_1^{\pm}\mathbf{t}^* + \tilde{\tilde{P}}_2^{\pm}\mathbf{r}_2^* + \tilde{\tilde{P}}_3^{\pm}\mathbf{r}_3^*, \quad \mathbf{\Phi}^{\pm} = \tilde{\mathbf{\Phi}}_1^{\pm}\mathbf{t}^* + \tilde{\mathbf{\Phi}}_2^{\pm}\mathbf{r}_2^* + \tilde{\mathbf{\Phi}}_3^{\pm}\mathbf{r}_3^* \tag{2.42}$$

must be considered specified.

To obtain consistent relations, expressions (2.22) and (2.23) must be introduced into Eqs (2.34) and (2.35), which, instead of Eq. (2.33), enables the following relations to be obtained

$$\begin{aligned}
&P_1^{\pm} = \underline{\tilde{\tilde{P}}_1^{\pm}} - \tilde{\tilde{P}}_2^{\pm}\chi + \tilde{\tilde{P}}_3^{\pm}\psi, \quad P_2^{\pm} = \tilde{\tilde{P}}_1^{\pm}(\upsilon' + ku) + \underline{\tilde{\tilde{P}}_2^{\pm}} - \tilde{\tilde{P}}_3^{\pm}\varphi, \quad P_3^{\pm} = \tilde{\tilde{P}}_1^{\pm}w' + \tilde{\tilde{P}}_2^{\pm}\varphi + \underline{\tilde{\tilde{P}}_3^{\pm}} \\
&\Phi_1^{\pm} = \underline{\tilde{\tilde{\Phi}}_1^{\pm}} - \tilde{\tilde{\Phi}}_2^{\pm}\chi + \tilde{\tilde{\Phi}}_3^{\pm}\psi, \quad \Phi_2^{\pm} = \tilde{\tilde{\Phi}}_1^{\pm}(\upsilon' + ku) + \underline{\tilde{\tilde{\Phi}}_2^{\pm}} - \tilde{\tilde{\Phi}}_3^{\pm}\varphi, \quad \Phi_3^{\pm} = \tilde{\tilde{\Phi}}_1^{\pm}w' + \tilde{\tilde{\Phi}}_2^{\pm}\varphi + \underline{\tilde{\tilde{\Phi}}_3^{\pm}}
\end{aligned} \tag{2.43}$$

3. The linearized equations of neutral equilibrium and their analysis

When expressions (2.6) are substituted into formulae (2.4) for the forces and moments entering Eq. (2.3), then Eqs (2.16) and conditions (2.17) prove to be extremely unwieldy. Their analysis shows that they contain both the "principal" terms present in well-known equations of flexible rod theory, derived by introducing greater constraints than those imposed above, and other ("non-principal") terms that are related, in particular, to the introduction of additional unknowns γ_2 and γ_3 to allow for transverse deformations of the rod. As follows from an analysis of results obtained earlier,[3,5] whether particular terms in relations (2.4) and (2.6) can be ignored depends mainly on the nature of external loading of the rod and on the type of stress state produced in it. Here, to estimate the degree to which the terms in expressions (2.4) are principal terms, one criterion can be the degree to which they influence the possibility of particular forms of loss of stability by the rod.

In this context, we will assume that, at a certain stage of loading of the rod, an initial stress state is formed characterized by the initial forces and moments $Q_x^0, Q_y^0, \ldots, M_{xz}^0, M_x^0 = M_{xz}^0 - M_{xy}^0$. If, in the vicinity of this equilibrium state, relations (2.4) are linearized by introducing the standard assumptions that the increments in the functions $u, \upsilon, w, \psi, \chi, \varphi, \gamma_2$ and γ_3 are small and equal to zero in the initial state, we obtain the following expressions for the increments in the forces and moments $Q_x^*, \ldots, N_z^*$

$$\begin{aligned}
Q_x^* &= Q_x - Q_y^0\chi + Q_z^0\psi \\
Q_y^* &= Q_y + Q_y^0\gamma_2 + Q_x^0(\upsilon' + ku) - Q_z^0\varphi - M_y^0(\varphi' - k\psi) - M_z^0(\gamma_2' - k\chi) \\
Q_z^* &= Q_z + Q_z^0\gamma_3 + Q_x^0 w' + Q_y^0\varphi + M_y^0\gamma_3' - M_z^0\varphi' \\
S_x^* &= k[Q_y + Q_y^0\gamma_2 + Q_x^0(\upsilon' + ku) - Q_z^0\varphi - M_y^0(\varphi' - k\psi) - M_z^0(\gamma_2' - k\chi)] \\
S_y^* &= -(Q_x - Q_y^0\chi + Q_z^0\psi)k \\
M_y^* &= M_y - M_{xy}^0\chi + S_{xz}^0\psi, \quad M_z^* = M_z - M_{xz}^0\psi + S_{xy}^0\chi \\
M_x^* &= M_x - M_{xy}^0\gamma_2 + M_{xz}^0\gamma_3 - M_y^0(\upsilon' + ku) - M_z^0 w' + (S_{xy}^0 + S_{xz}^0)\varphi \\
N_x^* &= M_y k + Q_y^0 w' - Q_z^0(\upsilon' + ku) + \\
&+ (T_y^0 + T_z^0)\varphi + kM_x^0\chi - M_{xz}^0(\gamma_2' - k\chi) + M_{xy}^0\gamma_3' + (S_{xy}^0 + S_{xz}^0)\varphi' \\
N_y^* &= -Q_y - kS_{xy} - Q_y^0(u' - k\upsilon) + M_z^0 k(\upsilon' + ku) + M_x^0 k\varphi - M_{xy}^0\psi' + T_y^0\chi + S_{xy}^0\chi' \\
N_z^* &= Q_z + M_{xy}k + Q_z^0(u' - k\upsilon) + M_y^0 k(\upsilon' + ku) - M_x^0 k\gamma_2 - M_{xz}^0\chi' + T_z^0\psi + S_{xz}^0\psi' \\
S_{xy}^* &= S_{xy} + S_{xy}^0\gamma_2 - M_z^0(\upsilon' + ku) - M_{xz}^0\varphi \\
S_{xz}^* &= S_{xz} + S_{xz}^0\gamma_3 + M_y^0 w' + M_{xy}^0\varphi \\
T_y^* &= T_y + M_z k + Q_y^0(\upsilon' + ku) - M_{xy}^0\varphi' - M_x^0 k\psi + S_{xy}^0\gamma_2' \\
T_z^* &= T_z + Q_z^0 w' + M_{xz}^0\varphi' + S_{xz}^0\gamma_3'
\end{aligned} \tag{3.1}$$

in which, unlike relations (2.6),

$$\begin{aligned}
Q_x &= F[g_{11}(u' - k\upsilon) + g_{12}\gamma_2 + g_{13}\gamma_3] \\
T_y &= F[g_{12}(u' - k\upsilon) + g_{22}\gamma_2 + g_{23}\gamma_3], \quad T_z = F[g_{13}(u' - k\upsilon) + g_{23}\gamma_2 + g_{33}\gamma_3] \\
M_y &= g_{11}J_y(\psi' + k\varphi), \quad M_z = g_{11}J_z(\chi' + k\gamma_2) \\
Q_y &= G_{12}G(\upsilon' + ku - \chi), \quad Q_z = G_{13}F(w' + \psi) \\
S_{xy} &= G_{12}J_z(\gamma_2' - k\chi), \quad S_{xz} = G_{13}J_y\gamma_3' \\
M_{xy} &= -G_{12}J_y(\varphi' - k\psi), \quad M_{xz} = G_{13}J_z\varphi'
\end{aligned} \tag{3.2}$$

Using the last two expressions in system (3.2), for the internal torque we obtain the relation

$$M_x = M_{xz} - M_{xy} = B_p\varphi' - G_{12}J_yk\psi \tag{3.3}$$

where $B_p = G_{13}J_z + G_{12}J_y$ is the torsional stiffness of the cross-section of a rod of orthotropic material, introduced into an earlier examination[5] of a straight rod.

If, for certain values of the “dead”[1] external forces applied to the rod, not only the initial equilibrium state but also a perturbed equilibrium state are possible, then, to determine their bifurcation values at which a transition from the initial state of equilibrium to the perturbed state occurs, use is made of the variational equation $\delta U = 0$, in which the expression for δU is identical in form to expression (2.3). From this equation, after standard conversions, we obtain a system of eight homogeneous differential equations of neutral equilibrium

$$\begin{aligned}
&\frac{dQ_x^*}{dx} - S_x^* = 0, \quad \frac{dQ_y^*}{dx} - S_y^* = 0, \quad \frac{dQ_z^*}{dx} = 0 \\
&\frac{dM_y^*}{dx} - N_z^* = 0, \quad \frac{dM_z^*}{dx} - N_y^* = 0, \quad \frac{dM_x^*}{dx} - N_x^* = 0 \\
&\frac{dS_{xz}^*}{dx} - T_z^* = 0, \quad \frac{dS_{xy}^*}{dx} - T_y^* = 0
\end{aligned} \tag{3.4}$$

and the static boundary conditions in the end cross-sections of the rod $x = x^-$ and $x = x^+$

$$\begin{aligned}
&Q_x^* = 0 \text{ when } \delta u \neq 0, \quad Q_y^* = 0 \text{ when } \delta\upsilon \neq 0, \quad Q_z^* = 0 \text{ when } \delta w \neq 0 \\
&M_y^* = 0 \text{ when } \delta\psi \neq 0, \quad M_z^* = 0 \text{ when } \delta\chi \neq 0, \quad M_x^* = 0 \text{ when } \delta\varphi \neq 0 \\
&S_{xy}^* = 0 \text{ when } \delta\gamma_2 \neq 0, \quad S_{xz}^* = 0 \text{ when } \delta\gamma_3 \neq 0
\end{aligned} \tag{3.5}$$

Using relations (3.1) and (3.2), Eqs (3.4) can be reduced to the form

$$\begin{aligned}
&L_{11}(w, \psi, \varphi) + [Q_z^0\gamma_3 + Q_x^0w' + Q_y^0\varphi + M_y^0\gamma_3' - M_z^0\varphi']' = 0 \\
&L_{12}(w, \psi, \varphi) - (M_{xy}^0\chi + S_{xz}^0\psi)' - Q_z^0(u' - k\upsilon) - M_y^0k(\upsilon' + ku) + \\
&+ M_x^0k\gamma_2 + M_{xz}^0\chi' - T_z^0\psi - S_{xz}^0\psi' = 0 \\
&L_{13}(w, \psi, \varphi) + [-M_{xy}^0\gamma_2 + M_{xz}^0\gamma_3 - M_y^0(\upsilon' + ku) - M_z^0w' + (S_{xy}^0 + S_{xz}^0)\varphi]' - \\
&- Q_y^0w' + Q_z^0(\upsilon' + ku) - (T_y^0 + T_z^0)\varphi + kM_x^0\chi + M_{xz}^0(\gamma_2' - k\chi) - M_{xy}^0\gamma_3' - (S_{xy}^0 + S_{xz}^0)\varphi' = 0 \\
&L_{21}(u, \upsilon, \chi, \gamma_2, \gamma_3) + [Q_y^0\gamma_2 + Q_x^0(\upsilon' + ku) - Q_z^0\varphi - M_y^0(\varphi' - k\psi) - M_z^0(\gamma_2' - k\chi)]' + \\
&+ k(-Q_y^0\chi + Q_z^0\psi) = 0
\end{aligned} \tag{3.6}$$

$$L_{22}(u, \upsilon, \chi, \gamma_2, \gamma_3) - (M_{xz}^0\psi - S_{xy}^0\chi)' + Q_y^0(u' - k\upsilon) - M_z^0 k(\upsilon' + ku) -$$
$$- M_x^0 k\varphi + M_{xy}^0\psi' - T_y^0\chi - S_{xy}^0\chi' = 0$$
$$L_{23}(u, \upsilon, \chi, \gamma_2, \gamma_3) + (-Q_y^0\chi + Q_z^0\psi)' - k[Q_y^0\gamma_2 + Q_x^0(\upsilon' + ku) -$$
$$- Q_z^0\varphi - M_y^0(\varphi' - k\psi) - M_z^0(\gamma_2' - k\chi)] = 0$$
$$L_{24}(u, \upsilon, \chi, \gamma_2, \gamma_3) + [S_{xy}^0\gamma_2 - M_z^0(\upsilon' + ku) - M_{xz}^0\varphi]' -$$
$$- Q_y^0(\upsilon' + ku) + M_{xy}^0\varphi' + M_x^0 k\psi - S_{xy}^0\gamma_2' = 0$$
$$L_{25}(u, \upsilon, \chi, \gamma_2, \gamma_3) + (S_{xz}^0\gamma_3 + M_y^0 w' + M_{xy}^0\varphi)' - Q_z^0 w' - M_{xz}^0\varphi' - S_{xz}^0\gamma_3' = 0 \tag{3.7}$$

where

$$L_{11}(w, \psi, \varphi) = [G_{13}F(w' + \psi)]'$$
$$L_{12}(w, \psi, \varphi) = [g_{11}J_y(\psi' + k\varphi)]' - G_{13}F(w' + \psi) + kG_{12}J_y(\varphi' - k\psi)$$
$$L_{13}(w, \psi, \varphi) = (B_p\varphi' - G_{12}J_y k\psi)' - g_{11}J_y k(\psi' + k\varphi)$$
$$L_{21}(u, \upsilon, \chi, \gamma_2, \gamma_3) = [G_{12}F(\upsilon' + ku - \chi)]' + kF[g_{11}(u' - k\upsilon) + g_{12}\gamma_2 + g_{13}\gamma_3]$$
$$L_{22}(u, \upsilon, \chi, \gamma_2, \gamma_3) = [g_{11}J_z(\chi' + k\gamma_2)]' + G_{12}F(\upsilon' + ku - \chi) + kG_{12}J_z(\gamma_2' - k\chi)$$
$$L_{23}(u, \upsilon, \chi, \gamma_2, \gamma_3) = \{F[g_{11}(u' - k\upsilon) + g_{12}\gamma_2 + g_{13}\gamma_3]\}' - kG_{12}F(\upsilon' + ku - \chi)$$
$$L_{24}(u, \upsilon, \chi, \gamma_2, \gamma_3) = [G_{12}J_z(\gamma_2' - k\chi)]' - F[g_{12}(u' - k\upsilon) + g_{22}\gamma_2 + g_{23}\gamma_3] - kg_{11}J_z(\chi' + k\gamma_2)$$
$$L_{25}(u, \upsilon, \chi, \gamma_2, \gamma_3) = (g_{13}J_y\gamma_3')' - F[g_{13}(u' - k\upsilon) + g_{23}\gamma_2 + g_{33}\gamma_3] \tag{3.8}$$

It can be seen that the differential expressions L_{1i} ($i = 1, 2, 3$) and L_{2j} ($j = 1, \ldots, 5$) contain different required functions. Therefore, when, as a result of the action of "dead" forces, only the initial forces and moments

$$Q_x^0 \neq 0, \quad M_z^0 \neq 0, \quad Q_y^0 \neq 0, \quad T_y^0 \neq 0, \quad T_z^0 \neq 0, \quad S_{xy}^{0'} \neq 0, \quad S_{xz}^{0'} \neq 0$$

are formed in the rod, and $Q_z^0 = M_y^0 = M_{xy}^0 = M_{xz}^0 = 0$ (consequently, $M_x^0 = M_{xz}^0 - M_{xy}^0 = 0$), systems (3.6) and (3.7) become isolated.

The first of these systems, which takes the form

$$L_{11} + (Q_x^0 w' + Q_y^0\varphi - M_z^0\varphi')' = 0, \quad L_{12} + (S_{xz}^0\psi)' - T_z^0\psi - S_{xz}^0\psi' = 0$$
$$L_{13} - [M_z^0 w' - (S_{xy}^0 + S_{xz}^0)\varphi]' - Q_y^0 w' - (T_y^0 + T_z^0)\varphi - (S_{xy}^0 + S_{xz}^0)\varphi' = 0 \tag{3.9}$$

describes the flexural-torsional FLS of the rod when $k \neq 0$, but, when T_z^0 is the only force formed, it describes the purely flexural FLS of a straight-axis rod.[5] Based on the most common version of the kinematic relations occurring under average bending, in Eqs (3.9) there remains only one parametric term $(Q_x^0 w')'$, which is the principal term. However, when $Q_x^0 = 0$, other parametric terms become the principal terms in Eqs (3.9), again leading to loss of stability of the rod. Some of these were studied earlier for a rectilinear rod.[5]

In the case considered, the second system of equations (3.7) takes the form

$$L_{21} + [Q_y^0\gamma_2 + Q_x^0(\upsilon' + ku) - M_z^0(\gamma_2' - k\chi)]' - kQ_y^0\chi = 0$$
$$L_{22} + (S_{xy}^0\chi)' + Q_y^0(u' - k\upsilon) - kM_z^0(\upsilon' + ku) - T_y^0\chi - S_{xy}^0\chi' = 0$$
$$L_{23} - (Q_y^0\chi)' - k[Q_y^0\gamma_2 + Q_x^0(\upsilon' + ku) - M_z^0(\gamma_2' - k\chi)] = 0$$
$$L_{24} + [S_{xy}^0\gamma_2 - M_z^0(\upsilon' + ku)]' - Q_y^0(\upsilon' + ku) - S_{xy}^0\gamma_2' = 0$$
$$L_{25} + (S_{xz}^0\gamma_3)' - S_{xz}\gamma_3' = 0 \tag{3.10}$$

Without loss of content or accuracy, the equations compiled allow considerable simplifications if the equations $S_{xz} = S_{xy} = 0$ are considered to hold. This is equivalent to determining the shear deformations by relations (1.11), in the first of which the term with the factor y is discarded, and in the second of which the term with the factor z is discarded. Here, from the last two equations of system (3.10), the functions γ_2 and γ_3 can easily be expressed in terms of the three remaining functions u, υ and χ. As a result of such transformations, we can obtain a separate system of equations for the functions u, υ and χ, in which the term $[Q_x^0(\upsilon' + ku)]'$ of the first equation is the main parametric term defining the flexural FLS in the direction of the y axis. The other FLS described by this system of equations for a rod with a rectilinear axis were studied earlier for certain special cases of loading.[5]

If the external forces are "follower" forces, then, after linearization, assuming that the rod is under stress but not deformed in the initial state, we obtain equations of neutral equilibrium of the same form as Eqs (2.16). The "loading" terms will be defined by the same formulae (2.11), in which the quantities $P_j^\pm$ and $\Phi_j^\pm$ are defined by linearized expressions (2.33) having the form

$$P_1^\pm = -\tilde{P}_1^\pm(\upsilon' + ku) - \tilde{P}_3^\pm w', \quad P_2^\pm = \tilde{P}_1^\pm(\upsilon' + ku) - \tilde{P}_3^\pm \varphi$$
$$P_3^\pm = \tilde{P}_1^\pm w' + \tilde{P}_2^\pm \varphi, \quad \Phi_1^\pm = -\tilde{\Phi}_2^\pm(\upsilon' + ku) - \tilde{\Phi}_3^\pm w'$$
$$\Phi_2^\pm = \tilde{\Phi}_1^\pm(\upsilon' + ku) - \tilde{\Phi}_3^\pm \varphi, \quad \Phi_3^\pm = \tilde{\Phi}_1^\pm w' + \tilde{\Phi}_2^\pm \varphi \tag{3.11}$$

if the external forces are specified by the components $\tilde{P}_j^\pm$ and $\tilde{\Phi}_j^\pm$. In this case, for the linearized equations boundary conditions (2.17) also retain their form, and here, unlike relations (2.9), the "loading" terms will be defined by the formulae

$$Q_x^s = -\tilde{Q}_y^s(\upsilon' + ku) - \tilde{Q}_z^s w', \quad Q_y^s = \tilde{Q}_x^s(\upsilon' + ku) - \tilde{Q}_z^s \varphi$$
$$Q_z^s = \tilde{Q}_x^s w' + \tilde{Q}_y^s \varphi, \quad M_y^s = -\tilde{M}_{xy}^s(\upsilon' + ku), \quad M_z^s = \tilde{M}_{xz}^s w'$$
$$M_x^s = -\tilde{M}_z^s w' - \tilde{M}_y^s(\upsilon' + ku)$$
$$M_{xy}^s = \tilde{M}_y^s(\upsilon' + ku), \quad M_{xz}^s = -\tilde{M}_z^s w' \tag{3.12}$$

In the case of when the "follower" external forces are specified by the components $\tilde{P}_j^\pm$ and $\tilde{\Phi}_j^\pm$, after linearization of expressions (2.43), the underlined terms disappear.

4. Forms of loss of stability of a circular ring for external pressure and compression in the radial direction

Suppose a circular ring of rectangular cross-section that has a centre-line radius $R = 1/k$, a width b and a thickness $2h$ is loaded with a pressure on its outer and inner surfaces $z = \pm h$ so that, for the initial stresses σ_{22}^0, the following boundary conditions are satisfied

$$\sigma_{22}^0(y = h) = -p - q, \quad \sigma_{22}^0(y = -h) = -q$$

With such a form of loading in a ring for which $2h \ll R$, the initial (subcritical) forces can be represented in the form

$$Q_x^0 = -RP, \quad T_y^0 = -Fq = -r_q RP = r_q Q_x^0, \quad P = pb \tag{4.1}$$

When the rod is exposed to external pressure p only, the relation $r_q \approx \varepsilon \ll 1$ holds. Since the remaining internal stresses and moments of the initial state are zero, when relations (3.8) and (4.1) are used the system of equations (3.9) takes the form

$$G_{13}F(w'' + \psi') - RPw'' = 0$$
$$g_{11}J_y(\psi'' + k\varphi') - G_{13}F(w' + \psi) + kG_{12}J_y(\varphi' - k\psi) = 0$$
$$B_p\varphi'' - kJ_y(G_{12} + g_{11})\psi' - k^2 g_{11}J_y\varphi + r_q RP\varphi = 0 \tag{4.2}$$

if the external forces p and q are "dead".

From the first equation in system (4.2), which describes the flexural-torsional FLS of a ring without linear deformation of its centre-line, the following relation is established after the integration constant has been neglected:

$$\psi = \left(\frac{RP}{G_{13}F} - 1\right)w' \tag{4.3}$$

When this relation is used, we obtain another relation from the second equation in system (4.2):

$$\varphi' = \frac{1}{k(g_{11} + G_{12})}\left\{-g_{11}\left(\frac{RP}{G_{13}F} - 1\right)w''' + \left[\frac{RP}{J_y} + k^2 G_{12}\left(\frac{RP}{G_{13}F} - 1\right)\right]w'\right\} \tag{4.4}$$

If the function w is now represented in the form

$$w = W_n \sin(ns/R) = W_n \sin n\theta, \quad \theta = s/R; \quad n = 0, 1, 2, \ldots \tag{4.5}$$

then, using relations (4.3) and (4.4), from the third equation in system (4.2), under the condition $W_n \neq 0$, we obtain the characteristic equation

$$\alpha r_q (RP)^2 - \beta_1 RP - \beta_2 r_q RP + \gamma = 0 \tag{4.6}$$

where

$$\alpha = \frac{G_{13}F + k^2G_{12}J_y}{kG_{13}J_yF} + \frac{g_{11}n^2k}{G_{13}F}$$

$$\beta_1 = \frac{1}{G_{13}F}\left[B_pg_{11}n^4k^3 + B_p\frac{G_{13}F + k^2G_{12}J_y}{J_y}n^2k - \right.$$

$$\left. -(2g_{11}G_{12} + G_{12}^2)J_yn^2k^3 + g_{11}(G_{13}F + k^2G_{12}J_y)k\right]$$

$$\beta_2 = k(g_{11}n^2 + G_{12})$$

$$\gamma = (B_pg_{11}n^4 + G_{12}G_{13}J_zn^2 - 2g_{11}G_{12}J_yn^2 + G_{12}g_{11}J_y)k^3 \tag{4.7}$$

we will first consider two special cases.

1°. Let $r_q \approx 0$, which corresponds to the classical formulation of the problem of the stability of a ring under an external pressure P, where the effect of the formation of the initial stress σ_{22}^0 in the radial direction in the rod on the bifurcation value P_* is ignored. In the case considered, from Eq. (4.6) we obtain the following formula for P_*

$$P^* = P_{ft}^* = \gamma/(\beta_1 R) \tag{4.8}$$

which, when $n = 0$, takes the form

$$P_{ft}^*(n=0) = \frac{G_{12}G_{13}FJ_yk^4}{G_{13}F + k^2G_{12}J_y} \tag{4.9}$$

The given bifurcation value of the load P, to which the solution with zero variability of the functions ψ, φ and w with respect to the circumferential coordinate of the ring corresponds, has no physical significance.

2°. Let $P = 0$ and $T_y^0 \neq 0$, which corresponds to simultaneous external and internal pressure on the ring in the radial direction. In the case considered, instead of Eq. (4.6), we obtain the equation $\beta_2 Fq = \gamma$, which yields the following expression for q^*

$$q_{ft}^* = \gamma/(\beta_2 F) \tag{4.10}$$

When $n = 0$, this takes the form

$$q_{ft}^*(n=0) = g_{11}J_yk^2/F \tag{4.11}$$

and, like formula (4.9), has no physical significance.When $P \neq 0$ and $T_y^0 \neq 0$, with a specified value of parameter r_q, the bifurcation values of the load P will be equal to the roots

$$P_{1,2}^{ft} = \frac{\beta_1 + r_q\beta_2 \pm \sqrt{(\beta_1 + r_q\beta_2)^2 - 4\alpha r_q\gamma}}{2\alpha r_q R} \tag{4.12}$$

of a quadratic equation, and the determination of its minimum positive value requires minimization of roots (4.12) with respect to the integer parameter n.

A second system of equations, describing the FLS of a rod in the sy plane, using formula (4.1) in approximation $S_{xy} = S_{zx} = 0$, takes the form

$$G_{12}G(\upsilon'' + ku' - \chi') - kF[g_{11}(u' - k\upsilon) + g_{12}\gamma_2 + g_{13}\gamma_3] - RP(\upsilon'' + ku') = 0$$

$$g_{11}J_z(\chi'' + k\gamma_2') + G_{12}F(\upsilon' + ku - \chi) + r_qRP\chi = 0$$

$$F[g_{11}(u'' - k\upsilon') + g_{12}\gamma_2' + g_{13}\gamma_3'] - kG_{12}F(\upsilon' + ku - \chi) + kRP(\upsilon' + ku) = 0$$

$$F[g_{12}(u' - k\upsilon) + g_{22}\gamma_2 + g_{23}\gamma_3] + kg_{11}J_z(\chi' + k\gamma_2) = 0$$

$$g_{13}(u' - k\upsilon) + g_{23}\gamma_2 + g_{33}\gamma_3 = 0 \tag{4.13}$$

For a thin-walled rod, with an accuracy $1 + \varepsilon^2 \approx 1$ we have the approximate equation

$$Fg_{22} + k^2J_zg_{11} \approx Fg_{22}$$

Therefore, from the last two equations in system (4.13) we obtain the relations

$$\gamma_2 = -\frac{1}{g_{22}g_{33}-g_{23}^2}\left[(g_{5-n\,5-n}g_{1n}-g_{1\,5-n}g_{23})(u'-kv)+\frac{kg_{11}g_{5-n\,3}J_z}{F}\chi'\right],\ n = 2, 3 \tag{4.14}$$

the use of which, with an accuracy $1+\varepsilon^2 \approx 1$, reduces system (4.13) to a system of three equations

$$\begin{aligned} &G_{12}F(v''+ku'-\chi')-kFE_1(u'-kv)-RP(v''+ku') = 0 \\ &g_{11}J_z\chi''+G_{12}F(v'+ku-\chi)+r_qRP\chi = 0 \\ &FE_1(u''-kv')-kG_{12}F(v'+ku-\chi)+kRP(v'+ku) = 0 \end{aligned} \tag{4.15}$$

where E_1 is the modulus of elasticity in the direction of the s axis.

For zero variability of the functions u, v and χ in the s direction, the first equation in system (4.15) has the solution $v=0$, while the remaining two equations take the form

$$\begin{aligned} &G_{12}Fku+(r_qRP-G_{12}F)\chi = 0 \\ &k^2(RP-G_{12}F)u+kG_{12}F\chi = 0 \end{aligned} \tag{4.16}$$

When $r_q=0$, under the conditions $\chi \neq 0$ and $u \neq 0$, the characteristic equation of system (4.16) leads only to the bifurcation value $P_*=0$, but with $P=0$, when $r_qRP=Fq$, we obtain the bifurcation value $q_*=0$.

Consequently, the adopted degree of accuracy of approximation of displacements in the form of Eq. (1.6) is insufficient to describe the purely shear FLS of the rod that occur, as established,[3,5] under the critical loads $P_s^* = FG_{12}/R$ and $q_s^* = G_{12}$. However, it should be noted that these solutions are of no interest in pratice, as the bifurcation values (4.8) and (4.10) corresponding to flexural-torsional FLS of the rod and also the plane flexural-shear FLS described by Eqs (4.15) with non-zero variability of the functions occurring in them are lower than the values for real rods. When the notation $\omega = v'+ku$ and $\varepsilon_1 = u'-kv$ is introduced, the first of these is rewritten in the form of the relation

$$FE_1\varepsilon_1 = -[RP\omega'-G_{12}F(\omega'-\chi')]/k \tag{4.17}$$

the use of which results in the third equation taking the form

$$-(RP-G_{12}F)\omega''-G_{12}F\chi''+k^2(RP-G_{12}F)\omega+k^2G_{12}F\chi = 0 \tag{4.18}$$

From the second equation of system (4.15) we obtain the relation

$$\omega = \left(1-\frac{r_qRP}{G_{12}F}\right)\chi-\frac{g_{11}J_z}{G_{12}F}\chi'' \tag{4.19}$$

substitution of which into Eq. (4.18) yields the resolvent of the problem

$$\left[RP(1+r_q)-\frac{r_q(RP)^2}{G_{12}F}-RP\frac{g_{11}J_z}{G_{12}F}\frac{d^2}{ds^2}+g_{11}J_z\frac{d^2}{ds^2}\right]\left(k^2-\frac{d^2}{ds^2}\right)\chi = 0 \tag{4.20}$$

Representing the function χ in the form

$$\frac{r_q}{G_{12}F}(PR)^2-RP(1+r_q+n^2k_{12})+g_{11}J_zn^2k^2 = 0;\quad k_{12} = \frac{g_{11}J_z}{G_{12}FR^2} \tag{4.21}$$

under the condition $\chi_n \neq 0$, from Eq. (4.20) we obtain the characteristic equation

$$\chi = \chi_n\sin n\theta,\quad \theta = s/R;\quad n = 0, 1, \ldots \tag{4.22}$$

In the general case, when non-zero forces P and q are both applied to the rod, the characteristic equation (4.18) can be represented in the form of Eq. (4.6), where

$$\alpha = 1/(G_{12}F),\quad \beta_1 = 1+n^2k_{12},\quad \beta_2 = 1,\quad \gamma = g_{11}J_zn^2k^2 \tag{4.23}$$

The roots of this equation are defined by formula (4.12), and from it, with $r_q=0$, we obtain the bifurcation value

$$P_*^{fs} = \frac{g_{11}J_zn^2}{(1+n^2k_{12})R^3} \tag{4.24}$$

and, with $P=0$ and $r_qRP=Fq$, for the bifurcation value we have

$$q_*^f = \frac{g_{11}J_zn^2}{FR^2} \tag{4.25}$$

The minimum P_* and q_* values for which flexural-shear and purely flexural FLS occur in the plane of the ring are obtained when $n=2$.

It should be noted that, as for a straight rod, formulae (4.24) and (4.25) correspond to the case where conservative forces are acting on the ring, retaining, on transition to the perturbed equilibrium state, the direction of the initial state. If forces P and q, causing the initial forces Q_x^0 and T_y^0 in the rod, are follower forces and directed during deformation along the normal to the centre-line, then the last two equations in system (4.15) take the form

$$g_{11}J_z\chi'' + G_{12}F(\upsilon' + ku - \chi) = 0$$
$$FE_1(u'' - k\upsilon') - kG_{12}F(\upsilon' + ku - \chi) = 0 \tag{4.26}$$

Here, the realization of only one flexural-shear FLS is possible, described by the resolvent

$$g_{11}J_z\left(1 - \frac{RP}{G_{12}F}\right)\chi^{\mathrm{IV}} + (RP + k^2 g_{11}J_z\chi'') = 0 \tag{4.27}$$

Its non-trivial solution is possible with a bifurcation value of the load P

$$P_*^{fs} = \frac{(n^2 - 1)g_{11}J_z}{(1 + n^2k_{12})R^3} \tag{4.28}$$

and here, as by formula (4.24), P_*^{fs} reaches a minimum value when $n = 2$.

Note that, in the formula derived, as in Eq. (4.24), the coefficient k_{12} appears because transverse shear has been taken into account. It differs from the classical formula in containing, instead of the modulus of elasticity E_1, the parameter

$$g_{11} = E_1(1 - \nu_{23}\nu_{32})/\Delta; \quad \Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{12}\nu_{23}\nu_{31}$$

which, in all the equations derived, appears naturally by reducing the initial three-dimensional problem to a one-dimensional problem. Since $q_{11} < E_1$, formula (4.28) leads to a lower value of P_* than the well-known classical formula

$$P_{\mathrm{cl}}^* = 3E_1J_z/R^3 \tag{4.29}$$

and the refined formula

$$P^* = \frac{3E_1J_z}{(1 + 4k_{12})R^3} \tag{4.30}$$

We will introduce the dimensionless parameter m of the load P, connecting them by the relation

$$P = m\frac{g_{11}J_z}{R^3} \tag{4.31}$$

as well as the dimensionless defining parameters

$$g_2 = \frac{G_{12}}{g_{11}}, \quad g_3 = \frac{G_{13}}{g_{11}}, \quad k_{13} = \frac{g_{11}J_z}{G_{13}FR^2}, \quad \delta = \frac{J_y}{J_z} \tag{4.32}$$

When they are used, Eq. (4.6) becomes

$$\tilde{\alpha}m^2 - \tilde{\beta}m + \tilde{\gamma} = 0 \tag{4.33}$$

where

$$\tilde{\alpha} = r_q b_1, \quad \tilde{\beta} = b_1[(g_3 + \delta g_2)n^2 + \delta] + b_2 r_q - \delta n^2 a_1 k_{13}(1 + g_2)$$
$$\tilde{\gamma} = b_2[(g_3 + \delta g_2)n^2 + \delta] - \delta n^2 a_1(1 + g_2)$$
$$a_1 = \delta k_{13}(g_2 + n), \quad b_1 = k_{13}n[1 + \delta k_{13}(g_2 + n^2)], \quad b_2 = n\delta k_{13}(g_2 + n^2) \tag{4.34}$$

When $r_q = 0$ ($q = 0$, $P \neq 0$), from relations (4.33) and (4.34) we obtain the bifurcation value of the parameter m, defined by the formulae

$$m_*^{ft} = \frac{b_2b_3 - b_4}{b_1b_3 - k_{13}b_4}, \quad b_3 = (g_3 + \delta g_2)n^2 + \delta, \quad b_4 = \delta a_1(1 + g_2)n^2 \tag{4.35}$$

This corresponds to a flexural-torsional FLS with bending of the ring in the direction of the binormal **b** (the z axis).

Our investigations indicate that the minimum values of m_*^{ft}, defined both by formula (4.35) when $r_q = 0$ and by the solution of Eq. (4.33) when $r_q \neq 0$, are obtained when $n = 2$, when the circular ring, on transferring to the perturbed state, is transformed into a "figure of eight". These values for a ring with the parameters $g_2 = g_3 = g$ are given in Table 1 for different values of g, k_{13}, δ and r_q. It can be seen that they are greater than $m_*^{ft} = 4$ (the classical solution of the problem of the plane flexural FLS of a ring according to the Bernoulli–Euler model under the action of a "dead" load P, when $r_q = 0$) only when $\delta = 2$ and $g = 0.38$ (an isotropic material) and for low values of r_q.

Consequently, to realize a plane flexural-shear FLS in a direction perpendicular to the plane of the ring (i.e., in the direction of the z axis), for the cross-section of the ring it is necessary to ensure a considerably greater bending stiffness than in the direction of the y axis,

Table 1

g	k_{13}	δ	$r_q=0$	0.001	0.01	0.1	1
0.38	0	0.5	1.599	1.598	1.595	1.564	1.169
		1	2.754	2.752	2.742	2.640	1.641
		2	4.754	4.751	4.725	4.462	1.596
	0.1	0.5	1.378	1.378	1.376	1.356	1.087
		1	2.159	2.158	2.153	2.014	1.512
		2	3.222	3.221	3.213	3.129	2.171
0.038	0	0.5	0.566	0.564	0.554	0.466	0.159
		1	0.793	0.791	0.774	0.629	0.201
		2	1.185	1.181	1.152	0.917	0.284
	0.1	0.5	0.535	0.534	0.526	0.449	0.158
		1	0.735	0.733	0.719	0.599	0.200
		2	1.059	1.057	1.036	0.856	0.282

especially when $g_2 \ll 1$ and $g_3 \ll 1$ and the ring possesses low torsional stiffness B_p. This conclusion is very basic, since, in the design of real frame structures, it is always ensured that the condition $\delta < 1$ is satisfied, assuming that, under the action of the load P, stability loss is in plane flexural form only. However, in this case the inequality

$$P_*^{fs} < P_*^{ft} = m_*^{ft} g_{11} J_z / R^3$$

which in current design practice is completely ignored, will always hold. True it must be pointed out that this conclusion applies only to isolated rings, since in real frame structures they are not isolated, and the possibility of stability loss of the forms described above is also determined by the stiffnesses of the other structural elements connected to them in the directions of the s, y and z axes.

Now suppose $P=0$ and $q \neq 0$. In this case, by introducing the parameter m_q, relating it to the load q by the equation

$$qF = m_q g_{11} J_z / R^3$$

from Eq. (4.6) we can obtain

$$m_q^* = \frac{(g_3 + \varepsilon g_2)n^2 + \varepsilon - \varepsilon^2 n^2 k_{13}(g_2 + n)(1 + g_2)}{\varepsilon k_{13} n (g_2 + n^2)}$$

It can be seen that $m_q^* \to 0$ as $n \to \infty$. Consequently, when the ring is compressed in a radial direction, flexural-shear FLS cannot occur. At the same time, as follows from Table 1, the effect of the load q on the critical value of the force P corresponding to this FLS is considerable at fairly high values of the parameter r_q.

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